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ON THE DEFORMATION OF AN ELASTIC HALF-SPACE WITH A THIN SLIT FOR MIXED CONDITIONS ON ITS BOUNDARY*

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The problem of the state of stress and strain in an elastic half-space with a cutout in the shape of a circular slot is solved by Kelvin's method /1/. The conditions on the slot are satisfied in a suitable manner by selecting the scalar and vector mass force potentials as generalized functions concentrated at the slot. The problem reduces to a system of Fredholm integral equations of the second kind in a semi-infinite interval. The solution for an elastic space with a slot is obtained in final form in the limiting case, which enables an estimate to be made of the magnitude of the settling of the earth's surface as a result of oil or gas deposit development.

1. Formulation of the problem. The axisymmetric problem of the stress and strain distribution in an elastic half-space E containing a cutout L in the form of an infinitely thin circular slot of radius R located parallel to the half-space boundary at a depth H is examined (see the sketch). A cylindrical r, z, θ system of coordinates is selected with origin at the centre of the slot, where the z-axis is directed towards the free surface perpendicular to it. The half-space boundary is stress-free while the displacements equal zero at infinity.

We start from the complete system of equations of the axisymmetric theory of elasticity that describe the state of strain of a body /2/

$$(\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{q} = 0 \tag{1.1}$$

$$\sigma = \frac{\lambda}{r}\frac{\partial}{\partial r}(ru_1) + (\lambda + 2\mu)\frac{\partial u_2}{\partial z}, \quad \tau = \mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r}\right)$$

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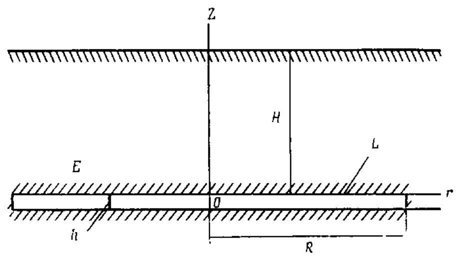
where λ, μ are the Lamé constants, $\mathbf{u}$ is the displacement vector, $u_1 = u_r$, $u_2 = u_z$, $\sigma = \sigma_z$ is the normal stress, and $\tau = \tau_{rz}$ is the shear stress. The mass force vector $\mathbf{q}$ is a generalized function concentrated on L (and equal to zero in $E \setminus L$) /3/. The boundary conditions on the free surface and at infinity are

$$\sigma = 0,\ \tau = 0,\ z = H;\ u_1 \to 0,\ u_2 \to 0,\ z \to -\infty \tag{1.2}$$

The set of all solutions of system (1.1) that satisfy conditions (1.2) can be expressed (by using integrals of the Hankel transformation type) in terms of two arbitrary functions of a single argument. Consequently, problem (1.1) and (1.2) reduces to a system of integral equations for these functions for any pair of independent boundary conditions on the slot surface.



Let us demonstrate this method by taking the problem with a given distribution of the shear stress τ and a linear combination of the normal stress σ and a jump in the axial displacement on the slot surface as a model

$$\tau = \tau_0(r),\ \sigma + \alpha[u_2] = \sigma_0(r),\ z = +0,\ r < R \tag{1.3}$$

where $\tau_0(r)$, $\sigma_0(r)$ are given continuous functions and $[u_2] = u_2(r;+0) - u_2(r;-0)$, α is a constant coefficient.

We will seek the set of quasiregular /4/ solutions of problem (1.1) and (1.2) by Kelvin's method /1/. Let $\mathbf{q} = \operatorname{grad}\varphi + \operatorname{rot}\mathbf{f}$, $\mathbf{f} = f\mathbf{e}_\theta$ (φ, $\mathbf{f}$ be scalar and vector potentials of the vector $\mathbf{q}$ and $\mathbf{e}_\theta$ the angular unit vector).

We set

$$\mathbf{u} = \operatorname{grad}\varphi_1 + \operatorname{rot}\mathbf{f}_1,\ \mathbf{f}_1 = f_1\mathbf{e}_\theta \tag{1.4}$$

where $\varphi_1(r,z)$, $f_1(r,z)$ are new unknown functions. Then /1/

$$(\lambda + 2\mu)\nabla^2\varphi_1 + \varphi = 0,\ \mu\nabla^2\mathbf{f}_1 + \mathbf{f} = 0$$

Since /5/ $\nabla^2\mathbf{f}_1 = (\nabla^2 f_1 - r^{-2}f_1)\mathbf{e}_\theta$ the system of equations to determine φ_1, f_1 (for given φ, f) takes the form (instead of (1.1))

$$(\lambda + 2\mu)\nabla^2\varphi_1 + \varphi = 0,\ \mu(\nabla^2 f_1 - r^{-2}f_1) + f = 0 \tag{1.5}$$

We apply a Hankel transformation with respect to the argument r. We use the notation Φ, Φ_1, U_2, S (functions of ξ and r) for the zero-order Hankel transforms for φ, φ_1, u_2, σ /2/. Analogously, let F, F_1, U_1, T be first-order Hankel transforms for f, f_1, u_1, τ. Applying the appropriate Hankel transforms to Eqs.(1.5), the second equation in (1.1) and (1.4) and the conditions (1.2), we obtain

$$\Phi_1'' - \xi^2\Phi_1 = -\Phi/(\lambda + 2\mu),\ F_1'' - \xi^2 F_1 = -F/\mu \tag{1.6}$$

$$S = \lambda\xi U_1 + (\lambda + 2\mu)U_2',\ T = \mu(U_1' - \xi U_2), \qquad U_1 = -(\xi\Phi_1 + F_1'),\ U_2 = \Phi_1' + \xi F_1 \tag{1.7}$$

$$S = 0,\ T = 0,\ z = H;\ U_1 \to 0,\ U_2 \to 0,\ z \to -\infty \tag{1.8}$$

(the prime denotes differentiation with respect to z in the space of generalized functions).

2. *Solution of the problem.* We set

$$\Phi = P_1(\xi)\delta(z) + \xi P_2(\xi)e^{-\xi|z|} + \xi P_3(\xi)e^{\xi z} \tag{2.1}$$

$$F = \xi P_2(\xi)e^{-\xi|z|}\,|z|' - \xi P_3(\xi)e^{\xi z}$$

($\delta(z)$ is the generalized delta function). It is easy to verify that the mass force $\mathbf{q} = 0$ in $E \setminus L$ when the necessary and sufficient conditions

$$\int_0^\infty P_i\xi^2 J_1(r\xi)\,d\xi = 0, \quad r > R, \quad i = 1, 2 \tag{2.2}$$

are satisfied.

The rest of the functions $P_i(\xi)$ $(i = 1, 2, 3)$ are as yet arbitrary. Substituting (2.1) into (1.6), finding the general solutions of these equations in the space of generalized functions, then determining U_1, U_2, S and T by means of (1.7) and satisfying conditions (1.8), we can express P_3 in terms of P_1 and P_2. Changing to dimensionless quantities by means of the formulas

$$t = R\xi, \quad \rho = \frac{r}{R}, \quad b = \frac{H}{R}, \quad P_1 + P_2 = \frac{\lambda + 2\mu}{\mu}\frac{p_0 R^2}{\xi} g_1(t) \tag{2.3}$$
$$\frac{\mu}{\lambda + 2\mu} P_1 + P_2 = \frac{p_0 R^2}{\xi} g_2(t), \quad \sigma_0^*(\rho) = \frac{\sigma_0}{p_0}, \quad \tau_0^*(\rho) = \frac{\tau_0}{p_0}$$

(p_0 is a characteristic constant having the dimensions of stress), the stress tensor and displacement vector components can be expressed in terms of two arbitrary functions $g_1(t)$ and $g_2(t)$ by using inverse Hankel transformation formulas. In particular

$$u_z(\rho; b) = -p_0 Q \int e^{-bt}((1 + 2bt)g_1 - btg_2)J_0(\rho t)dt \tag{2.4}$$
$$Q = R(\lambda + 2\mu)\mu^{-1}(\lambda + \mu)^{-1}$$

(Here and henceforth, integration over t is between 0 and $+\infty$). Satisfying conditions (1.3) we obtain by taking account of (2.3)

$$\int tg_1 J_0(\rho t)\,dt = \int te^{-2bt}(A^+(t)\,g_1 - B(t)\,g_2)\,J_0(\rho t)\,dt + \tag{2.5}$$
$$\alpha Q \int \left(g_1 - \frac{\mu}{\lambda + 2\mu} g_2\right) J_0(\rho t)\,dt + \sigma_0^*(\rho), \quad \rho < 1; \quad \int g_1 J_0(\rho t)\,dt = 0, \quad \rho > 1$$

$$\int tg_2 J_1(\rho t)dt = \int te^{-2bt}(A^-(t)g_2 - B(t)g_1)J_1(\rho t)dt + \tag{2.6}$$
$$\tau_0^*(\rho), \ \rho < 1; \quad \int tg_2 J_1(\rho t)dt = 0, \ \rho > 1$$
$$A^\pm(t) = 1 \pm 2bt + 2b^2t^2, \ B(t) = 2b^2t^2$$

A system of integral equations for the functions $g_1(t)$ and $g_2(t)$ results.

3. ***A special case of an infinite space with given shear and normal stresses on the slot surface.*** For $H \to +\infty$ $(b \to +\infty)$, $\alpha = 0$ system (2.5) and (2.6) simplifies to

$$\int tg_1 J_0(\rho t)dt = \sigma_0^*(\rho), \ \rho < 1; \quad \int g_1 J_0(\rho t)\,dt = 0, \ \rho > 1 \tag{3.1}$$
$$\int tg_2 J_1(\rho t)dt = \tau_0^*(\rho), \ \rho < 1; \quad \int tg_2 J_1(\rho t)\,dt = 0, \ \rho > 1 \tag{3.2}$$

The system of dual Eqs.(3.1) is solved in /2/

$$g_1(t) = \frac{2}{\pi}\int_0^1 x \sin tx\,dx \int_0^1 \frac{\rho\sigma_0^*(\rho x)}{\sqrt{1 - \rho^2}}\,d\rho \tag{3.3}$$

System (3.2) is solved by Hankel inversion formulas

$$g_2(t) = \int_0^1 \tau_0^*(\rho)\,\rho J_1(t\rho)\,d\rho \tag{3.4}$$

If $\sigma_0(r) \equiv p_0$, $\tau_0(r) \equiv \tau_0$ (p_0, τ_0 are given constants), then according to (3.3) and (3.4)

$$g_1(t) = -\frac{}{\pi}\frac{d}{dt}\left(\frac{\sin t}{t}\right), \quad g_2(t) = \frac{\tau_0}{p_0}\int_0^1 xJ_1(tx)\,dx \tag{3.5}$$

Finding the formula

$$u_z(\rho; +0) = -\frac{1}{2} p_0 Q \int (g_1 - \mu(\lambda + 2\mu)^{-1} g_2) J_0(\rho t)\, dt$$

as $H \to +\infty$ substituting (3.5), and simplifying, we find

$$u_z(\rho; +0) = \pi^{-1} p_0 Q \sqrt{1-\rho^2} + \frac{1}{2}\tau_0 R (\lambda+\mu)^{-1}(1-\rho), \quad \rho < 1$$

For $\tau_0 = 0$ this formula reduces to that known in /2/.

4. The general case. Reduction to a system of Fredholm integral equations. The system of two Eqs.(2.5) can be reduced to one equation by the method by which system (3.1) was solved. By understanding σ_0^* in (3.3) to be the whole right-hand side of the first equation in (2.5), changing the order of integration, and simplifying, we can obtain

$$g_1(t) = \frac{2}{\pi}\int_0^1 x \sin tx\, dx \int_0^1 \frac{\rho\sigma_0^*(\rho x)}{\sqrt{1-\rho^2}}\, d\rho + \tag{4.1}$$

$$\frac{2}{\pi}\int_0^\infty \frac{x \sin t \cos x - t \sin x \cos t}{t(t^2 - x^2)} [te^{-2bx}(A^+(x) g_1(x) - B(x) g_2(x)) -$$

$$\alpha Q (g_1(x) - \mu(\lambda + 2\mu)^{-1} g_2(x))]\, dx$$

Analogously, from the second equation in (2.5) we have

$$g_2(t) = \int_0^1 \tau_0^*(\rho)\rho J_1(t\rho)\, d\rho + \int_0^\infty \frac{tJ_1(t) J_0(x) - xJ_0(t) J_1(x)}{t^2 - x^2} e^{-2bx}(A^-(x) g_2(x) - \tag{4.2}$$

$$B(x) g_1(x))\, dx$$

Therefore, the solution of the problem reduces to the problem of integrating a system of two Fredholm integral equations of the second kind with continuous kernels, where the desired functions are $g_1(t),\ g_2(t) \in C\,[0; +\infty[$.

5. Application to the problem of the settling of the bottom surface during oil and gas deposit development. When oil and gas deposits having a stratal pressure that varies with time are developed, the deformation of the mountain rocks reaches the bottom surface causing it to subside. Consequently, the development of a method for making quantitative predictive estimates of this deformation is quite important.

As a result of the development of a certain deposit let the stratal pressure change by an amount Δp after which an equilibrium state occurs in the stratum and mountain rocks. Because of the comparatively small deformations it can be assumed that the rocks surrounding the stratum are deformed linerly elastically. The stratum has the shape of a thin circular cylinder of radius R and thickness h, where $h \ll R$, so that the stratum is replaced by a slot of radius R (see the sketch). The stresses σ and τ equal zero on the bottom surface ($z = H$).

To a first approximation we set $\tau = 0$ in the rock near the stratum while we express the normal stress σ in terms of the displacement by taking the following scheme for deformation of the rocks of the stratum. The stratum is assumed to comprise granular or cracked rocks with extensively developed fracturing. A vertical component of the mountain pressure σ_1 acts on the stratum elements, the stress equals σ_2 in the rock skeleton, and fluid or gas with pressure p is in the pore space where

$$\sigma_1 = \sigma_2 + p, \quad \Delta\sigma_2 = \Delta\sigma_1 - \Delta p \tag{5.1}$$

The volume of a cylindrical ring of height h with inner radius r and outer radius $r + dr$ equals $V = 2\pi rh dr$ its change (if radial displacements are neglected) is $\Delta V = 2\pi r dr\,[u_2]$. Therefore $\Delta V/V = [u_2]/h$. On the other hand $\Delta V/V = m_0\beta_2\Delta\sigma_2 + (1 - m_0)\beta\Delta p$ where m_0 is the initial porosity of the stratum and β_2, β is the compressibility factor of the pore and the skeleton. Eliminating $\Delta V/V$ and $\Delta\sigma_1$ from these relationships by using (5.1) we obtain for $z = 0$ by setting $\sigma = \Delta\sigma_1$

$$\sigma - \frac{[u_2]}{hm_0\beta_2} = p_0, \quad z = 0, \quad r < R; \quad p_0 = \left(1 - \frac{(1-m_0)\beta}{m_0\beta_2}\right)\Delta p$$

Therefore, conditions (1.3) with $\sigma_0(r) \equiv p_0$, $\tau_0(r) \equiv 0$, $\alpha = -(hm_0\beta_2)^{-1}$ occur.

We consider the quantities m_0, β, β_2, Δp constants. The problem reduces to solving system (2.5) and (2.6) with $\tau_0^* \equiv 0$, $\sigma_0^* \equiv 1$.

For the upper limit of the maximum deflection (i.e., for $r = 0$) of the bottom surface we note that it will be greater the smaller the value of H (i.e., the smaller the value of b). It can be shown by analysing (2.5) and (2.6) (with $\tau_0^* \equiv 0, \sigma_0^* \equiv 1$) that

$$g_2 \to 0, \int g_1 J_0(\rho t) dt \to -hm_0\beta_2 Q^{-1}, \ \rho < 1, \ b \to 0$$

Substituting into (2.4), we obtain as $b \to 0$

$$u_z(r; H) = p_0 h m_0 \beta(\lambda + 2\mu)(\lambda + \mu)^{-1}, \ r < R$$

Thus, for any H (i.e., for any b)

$$|u_z(r; H)| < 2(1 - \nu)|p_0| h m_0 \beta_2, \ \nu = {}^1/_2 \lambda (\lambda + \mu)^{-1} \tag{5.2}$$

For numerical data /6/ $\Delta p = -40$ MPa, $h = 600$ m, $R = 10^4$ m, $\beta_2 = 2 \times 10^{-3}$ (MPa)$^{-1}$, $\beta = 1.5 \times 10^{-5}$ (MPa)$^{-1}$, $m_0 = 0.05, \nu = 0.34$ we obtain $p_0 = -34.3$ MPa, from which $|u_z(r; H)| < 2.74$ m according to (5.2), which agrees with the approximate estimate obtained in /7/.

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A GEOMETRICAL METHOD OF SOLVING THE PROBLEM OF MAXIMIZING THE NORM OF THE STATE VECTOR OF THE SYSTEM IN A FINITE CONTROL INTERVAL*

A.M. TKACHEV

The problem of constructing controls which maximize the norm of the state vector of the system at the right-hand end of a fixed control interval is considered. A numerical method of determining the maxima is proposed, based on a geometrical approach. Local convergence of the algorithm is proved and the direction of the search for the global maximum is discussed. Results of numerical modelling are given.

The problem of maximizing the convex function J on a convex manifold of attainability discussed here, cannot be solved using traditional methods (for example, the method of minimum discrepancy and its modifications /1, 2/), since in the case of an equivalent minimization the functional J is not convex. This leads, in particular, to violation of the theorems of uniqueness of optimal control. Indeed (Fig.1), more than one point may exist belonging to the convex manifold of attainability $K(T)$ at the maximum distance from the origin of coordinates. At the same time, there exists a unique point belonging to $K(T)$ whose distance from the origin of coordinates is a minimum.

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